

CLASSIFICATION OF 7-DIMENSIONAL EINSTEIN NILRADICALS

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ABSTRACT. The problem of classifying Einstein solvmanifolds, or equivalently, Ricci soliton nilmanifolds, is known to be equivalent to a question on the variety $\mathfrak{N}_n(\mathbb{C})$ of n -dimensional complex nilpotent Lie algebra laws. Namely, one has to determine which $\mathrm{GL}_n(\mathbb{C})$ -orbits in $\mathfrak{N}_n(\mathbb{C})$ have a critical point of the squared norm of the moment map. The set $\mathfrak{N}_7(\mathbb{C})/\mathrm{GL}_7(\mathbb{C})$ is formed by 148 nilpotent Lie algebras and 6 curves of pairwise non-isomorphic nilpotent Lie algebras. In this paper, we give a complete classification of the aforementioned distinguished orbits for $n = 7$.

1. INTRODUCTION

From general theory of relativity, a Riemannian manifold (M, g) is said to be *Einstein* if its Ricci tensor complies the Einstein condition $\mathrm{ric} = cg$ for some constant $c \in \mathbb{R}$ and in such case, g is called an *Einstein metric*. If M is compact, Einstein metrics are identified as critical points of the total scalar curvature functional on the space of all metrics on M of a fixed volume, and can therefore be considered as “selected elements”.

At the present time, no general existence results for Einstein metrics are known. Even in the homogeneous case, $M = G/H$, the search for G -invariant Einstein metrics (G -invariance turns the problem into an “algebraic” one) is a difficult problem. We are interested in homogeneous Einstein metrics of negative scalar curvature (noncompact, nonflat). A big challenge in this case is known as the *Alekseevskii conjecture*, which states that any homogeneous Einstein manifold with negative scalar curvature is isometric to an *Einstein solvmanifold*, i.e. a simply connected solvable Lie group endowed with a left-invariant metric satisfying the Einstein condition (see [B, Conjecture 7.57]).

In [H], Heber gives a detailed analysis of standard Einstein solvmanifolds, this means that the corresponding metric solvable Lie algebra $(\mathfrak{s}, \langle \cdot, \cdot \rangle)$ has orthogonal decomposition $\mathfrak{s} = \mathfrak{n} \oplus^\perp \mathfrak{a}$ with $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ and $[\mathfrak{a}, \mathfrak{a}] = 0$. Heber’s paper is accepted by many researchers as a pioneering work and one of his results allows Einstein solvmanifolds to be studied with geometric invariant

2000 *Mathematics Subject Classification.* Primary 53C25; Secondary 53C30, 22E25.

Key words and phrases. Einstein manifolds, Einstein Nilradical, Nilsolitons, Geometric Invariant Theory, Nilpotent Lie Algebras.

Fully supported by a CONICET fellowship (Argentina).

theory (GIT). Since then, geometric invariant theory has played an important role in understanding Einstein solvmanifolds and recent advances have come by this powerful machinery. A sample of this is the next result due to Lauret, in which the Kirwan's stratification of the variety of nilpotent Lie algebra laws is in the core of the proof.

Theorem 1.1. [L4, Theorem 3.1] *Any Einstein solvmanifold is standard.*

In [Nk2], Nikolayevsky gives some structural results on Einstein nilradicals, as well as criteria to decide if a nilpotent Lie algebra is Einstein nilradical or not. An *Einstein nilradical* is a nilpotent Lie algebra which is the nilradical of an Einstein solvmanifold.

There are many equivalent conditions to be an Einstein nilradical, some of them related with the Ricci Flow. Let N be the simply connected nilpotent Lie group with Lie algebra \mathfrak{n} . Then $(S, \langle \cdot, \cdot \rangle)$ is Einstein if and only if $(N, \langle \cdot, \cdot \rangle|_{\mathfrak{n}})$ is a *Ricci soliton* (called nilsolitons in the literature, cf. [L3]), i.e. the Ricci flow solution starting at $(N, \langle \cdot, \cdot \rangle|_{\mathfrak{n}})$ only evolves by pullback of diffeomorphisms and scaling. Furthermore, the only known examples of nontrivial homogeneous Ricci solitons are all solvmanifolds which can be constructed as a semidirect product of an Einstein nilradical and a suitable torus of automorphisms (see [L5]).

Our main result in this paper is a complete classification of 7-dimensional nilpotent Lie algebras which are Einstein nilradicals. Since any nilpotent Lie algebra of dimension less or equal than 6 is an Einstein nilradical ([L2], [W]) and a direct sum of Einstein nilradicals is again an Einstein nilradical, we focus on studying indecomposable algebras. By [Nk2, Theorem 6], it is enough to consider complex nilpotent Lie algebras and this is why we use the classification in dimension 7 given in [C] (with a couple of corrections by Magnin in [M]), which consists of a list of 117 non-isomorphic indecomposable algebras and 6 curves of pairwise non-isomorphic nilpotent Lie algebras. If a nilpotent Lie algebra is written in a *nice basis* (see Definition 2.7), then by using [Nk2, Theorem 3] it is a simple matter to prove whether the algebra is Einstein nilradical or not. In [M], 20 algebras and one curve are not written in a nice basis, so that these are studied by other methods. Namely,

- by exhibiting a nilsoliton inner product (see Corollary 2.3).
- by studying the closeness of the orbit of a nilpotent Lie algebra law by the action of certain reductive algebraic subgroup of $GL_n(\mathbb{R})$ on $\Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ (cf. [Nk2, Theorem 5]).

The first rank-zero nilpotent Lie algebras (also known as characteristically nilpotent Lie algebras) appear in dimension 7, which can not be Einstein nilradicals since they do not admit an \mathbb{N} -gradation. This family has 7 nilpotent Lie algebras and one curve. For rank ≥ 1 , we get 82 indecomposable Einstein nilradicals (from a total of 110 indecomposable algebras) and 5 curves of Einstein nilradicals (with the exception of at most 2 points in

each curve; recall that to be an Einstein nilradical is not a property which depends continuously on the structural constants of the Lie algebra).

It is known that the variety of n -dimensional complex nilpotent Lie algebra laws, $\mathfrak{N}_n(\mathbb{C})$, is reducible for any $n \geq 7$ (cf. [KAG]). For the natural action of $\mathrm{GL}_n(\mathbb{C})$ on $\Lambda^2(\mathbb{C}^n)^* \otimes \mathbb{C}^n$, by applying Kirwan's stratification and convexity properties of the moment map, it follows that each irreducible component \mathcal{C}_i of $\mathfrak{N}_n(\mathbb{C})$ can be obtained as the closure of its unique stratum of minimum norm (cf. [F2]). There are finitely many strata and are parameterized by the moment map images of the critical points of the squared norm of the moment map that belong to $\mathfrak{N}_n(\mathbb{C})$ (which are precisely Einstein nilradicals). It follows from the classification of 7-dimensional Einstein nilradicals obtained in the present paper that the two irreducible components of $\mathfrak{N}_7(\mathbb{C})$ are respectively the closures of the strata corresponding to the Einstein nilradicals of eigenvalue type $(1 < 2 < 3 < 4 < 5; 2, 1, 2, 1, 1)$ and $(1 < 2 < 3 < 4 < 5 < 6 < 7; 1, \dots, 1)$. We note that the stratum associated to the first type above has actually minimal norm over all strata of $\mathfrak{N}_7(\mathbb{C})$, although a third stratum has smaller norm than the second one, the strata attached to the type $(16 < 21 < 37 < 48 < 53 < 69 < 90; 1, \dots, 1)$ (see Tables 1 and 2). A characterization of the strata of minimum norm in the irreducible components of $\mathfrak{N}_n(\mathbb{C})$ for n large, may be very useful in the study of many problems on the variety of Lie algebra laws, as for instance rigidity of nilpotent Lie algebras (a conjecture attributed to Michèle Vergne says that for n sufficiently large, there do not exist nilpotent Lie algebras which have Zariski-open orbit in $\mathfrak{N}_n(\mathbb{C})$).

2. PRELIMINARIES

In this section, we overview all the results on Einstein nilradicals we need to apply in our classification. We refer to the survey [L3] for a more detailed treatment.

It follows from Theorem 1.1 and Heber's Rank-One reduction ([H, Theorem 4.18]) that any Einstein solvmanifold with Einstein metric solvable Lie algebra $(\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}, \langle \cdot, \cdot \rangle)$ is determined by the Einstein metric solvable Lie algebra $(\mathfrak{n} \rtimes \mathbb{R}\phi, \langle \langle \cdot, \cdot \rangle \rangle)$ where $\langle \langle \cdot, \cdot \rangle \rangle$ denotes the restriction of the inner product on \mathfrak{s} and ϕ is a special derivation with positive integer eigenvalues (it is called *Einstein derivation* in [Nk2, Definition 1]). Since the Einstein derivation and $\langle \langle \cdot, \cdot \rangle \rangle$ only depend on the nilpotent Lie algebra \mathfrak{n} (see [L1, Lemma 2]), the classification of Einstein solvmanifolds is a problem on nilpotent Lie algebras.

Definition 2.1. A nilpotent Lie algebra which is the nilradical of an Einstein metric solvable Lie algebra is called an *Einstein nilradical*.

Let V be the vector space $\Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$. If $\{e'_1, \dots, e'_n\}$ is the basis of $(\mathbb{R}^n)^*$ dual to the canonical basis $\{e_1, \dots, e_n\}$ of \mathbb{R}^n then $\mu_{ij}^k = (e'_i \wedge e'_j) \otimes e_k$ with $1 \leq i < j \leq n$, $1 \leq k \leq n$ is the canonical basis of V . As we mentioned above, GIT is a very important tool in the theory of Einstein solvmanifolds

and the connection is the moment map for the natural action of $\mathrm{GL}_n(\mathbb{R})$ on V , $\tilde{m} : V \setminus \{0\} \rightarrow \mathrm{sym}(n)$. Let $\mu \in V$ be a nilpotent Lie algebra law and let Ric_μ denote the Ricci operator of the nilmanifold $(N_\mu, \langle \cdot, \cdot \rangle)$, where N_μ is the simply connected Lie group with $\mathrm{Lie}(N_\mu) = (\mathbb{R}^n, \mu)$ and $\langle \cdot, \cdot \rangle$ is the canonical inner product of \mathbb{R}^n . For those who are familiar with the works of Ness ([N]), Kirwan ([Kr]) or Marian ([Mr]), it will be enough to know that

$$(2.1) \quad \tilde{m}(\mu) = \frac{4}{\|\mu\|^2} \mathrm{Ric}_\mu,$$

to recognize some of the results about existence and uniqueness of Einstein solvmanifolds (Einstein nilradical).

Let m be the unnormalized moment map, given by $m(\mu) = 4 \mathrm{Ric}_\mu$. By closeness with GIT, we work with the moment map m instead of Ric . Computations in [F1] are made using m .

The next theorem gives an characterization of Einstein nilradicals.

Theorem 2.2. [L3, Theorem 4.2.] *Let $\mu \in V$ be a nilpotent Lie algebra law. The nilpotent Lie algebra (\mathbb{R}^n, μ) is an Einstein nilradical if and only if there exists $\tilde{\mu} \in \mathrm{GL}_n(\mathbb{R}) \cdot \mu$ such that $\mathrm{Ric}_{\tilde{\mu}} \in \mathbb{R}I \oplus \mathrm{Der}(\tilde{\mu})$, or equivalently, $m(\tilde{\mu}) = c_{\tilde{\mu}}I + \phi$ for some $c_{\tilde{\mu}} < 0$ and ϕ a symmetric derivation of $\tilde{\mu}$.*

The derivation ϕ in Theorem 2.2 is, up to conjugation, a positive multiple of the Einstein derivation. Recall that the eigenvalues of the Einstein derivation are positive integer and define the *eigenvalue type* (see [L3, Definition 2.6.]).

Corollary 2.3. *Let $\mu \in V$ be a nilpotent Lie algebra law. The nilpotent Lie algebra (\mathbb{R}^n, μ) is an Einstein nilradical with eigenvalue type $(d_1 < \dots < d_r; n_1, \dots, n_r)$ if and only if there exists $\tilde{\mu} \in \mathrm{GL}_n(\mathbb{R}) \cdot \mu$ such that:*

$$(2.2) \quad m(\tilde{\mu}) = \frac{\sum n_i d_i}{n \sum n_i d_i^2 - (\sum n_i d_i)^2} \left(-\frac{\sum n_i d_i^2}{\sum n_i d_i} I + \mathrm{Diag} \left(\underbrace{d_1}_{n_1 \text{ times}}, \dots, \underbrace{d_r}_{n_r \text{ times}} \right) \right)$$

where d_i , $i = 1, \dots, r$ are positive integers without a common divisor and $\mathrm{Diag}(d_1, \dots, d_r)$ is a derivation of $\tilde{\mu}$.

Proof. Let $\mu_0 \in \mathrm{GL}_n(\mathbb{R}) \cdot \mu$ such that $m(\mu_0) = c_{\mu_0}I + \phi$. Since the matrix of $m(\mu_0)$ in the canonical basis of \mathbb{R}^n is symmetric, then there exists $k \in \mathrm{O}_n(\mathbb{R})$ such that $k m(\mu_0) k^{-1} = \mathrm{Diag}(x_1, \dots, x_n)$ with $x_1 \leq \dots \leq x_n$ and so $k \phi k^{-1} = a \mathrm{Diag}(d_1, \dots, d_r)$, where $d_1 < \dots < d_r$ are the eigenvalues of the Einstein derivation ϕ with multiplicities n_1, \dots, n_r . By using that the moment map is $\mathrm{O}_n(\mathbb{R})$ -equivariant we have

$$(2.3) \quad m(k \cdot \mu_0) = c_{\mu_0}I + a \mathrm{Diag}(d_1, \dots, d_r).$$

Now, since the moment map is orthogonal to each symmetric derivation (by [L3, Equation (4.4)]), if we take the inner product of equation (2.3) with

$\text{Diag}(d_1, \dots, d_r)$, then we get that

$$a = -c_{\mu_0} \frac{\sum n_i d_i}{\sum n_i d_i^2}$$

and therefore, by replacing a in (2.3)

$$(2.4) \quad m(k \cdot \mu_0) = c_{\mu_0} \left(I - \frac{\sum n_i d_i}{\sum n_i d_i^2} \text{Diag}(d_1, \dots, d_r) \right)$$

We recall that, for any $\lambda \in V$, $\text{tr}(m(\lambda)) = -\|\lambda\|^2$ (again, by [L3, Equation 4.4.] with $\alpha = I$), therefore, if we use this fact in (2.4), it follows that

$$c_{\mu_0} = -\|c_{\mu_0}\|^2 / \left(n - \frac{(\sum n_i d_i)^2}{\sum n_i d_i^2} \right)$$

the denominator above is not zero by the CauchySchwarz inequality and $\phi \notin \mathbb{R}I$. In consequence

$$\frac{m(k \cdot \mu_0)}{\|k \cdot \mu_0\|^2} = m \left(\frac{k \cdot \mu_0}{\|k \cdot \mu_0\|} \right) = (2.2)$$

and, as $\frac{k \cdot \mu_0}{\|k \cdot \mu_0\|} \in \text{GL}_n(\mathbb{R}) \cdot \mu$, making $\tilde{\mu} = \frac{k \cdot \mu_0}{\|k \cdot \mu_0\|}$ we have concluded the proof. \square

The importance of expression (2.2) is that it only depends on the eigenvalue type. We use the pre-Einstein derivation and Corollary 2.3 to find Einstein nilradicals of a fixed eigenvalue type.

If (\mathbb{R}^n, μ) is an Einstein nilradical with eigenvalue type $(d_1 < \dots < d_r; n_1, \dots, n_r)$, then the value

$$(2.5) \quad \left(n - \frac{(\sum n_i d_i)^2}{\sum n_i d_i^2} \right)^{-1}$$

is the minimum value on $\text{GL}_n(\mathbb{R}) \cdot \mu$ that the function

$$(2.6) \quad \begin{array}{ccc} F : V \setminus \{0\} & \longrightarrow & \mathbb{R} \\ v & \longmapsto & \|\tilde{m}(v)\|^2 \end{array}$$

takes. F measures how far the Einstein derivation is from a multiple of the identity. Since there are finitely many eigenvalue types (or equivalently, there are finitely many *strata*), then the expression (2.5) takes finitely many values which can be used to study degenerations in $\mathfrak{N}_n(\mathbb{R})$ via the stratification ([L4, Theorem 2.10.]).

To finish, we review some definitions and results from [Nk2].

Definition 2.4. [Nk2, Definition 2] A derivation ϕ of a Lie algebra \mathfrak{g} is called *pre-Einstein*, if it is semisimple, with all the eigenvalues real, and

$$(2.7) \quad \text{tr}(\phi\psi) = \text{tr}(\psi), \text{ for any } \psi \in \text{Der}(\mathfrak{g})$$

Theorem 2.5. [Nk2, Theorem 1]

1. (a) Any Lie algebra \mathfrak{g} admits a pre-Einstein derivation $\phi_{\mathfrak{g}}$.

- (b) The derivation $\phi_{\mathfrak{g}}$ is determined uniquely up to automorphism of \mathfrak{g} .
- (c) All the eigenvalues of $\phi_{\mathfrak{g}}$ are rational numbers.
- 2. Let \mathfrak{n} be a nilpotent Lie algebra, with ϕ a pre-Einstein derivation. If \mathfrak{n} is an Einstein nilradical, then its Einstein derivation is, up to conjugation by an automorphism, positively proportional to ϕ and

$$(2.8) \quad \phi > 0 \text{ and } \text{ad}_{\phi} \geq 0,$$

i.e. all eigenvalues of ϕ are positive and all eigenvalues of ad_{ϕ} are non negative.

There is a reductive real Lie group, G_{ϕ} , attached to a pre-Einstein derivation. Consider

$$(2.9) \quad \mathfrak{g}_{\phi} := \{\alpha \in \mathfrak{gl}_n(\mathbb{R}) : [\alpha, \phi] = 0, \text{tr}(\alpha\phi) = 0, \text{tr}(\alpha) = 0\}$$

and let G_{ϕ} be the connected Lie subgroup of $\text{GL}_n(\mathbb{R})$ with Lie algebra \mathfrak{g}_{ϕ} . The relevance of the group G_{ϕ} in the study of Einstein nilradicals is given by the next theorem.

Theorem 2.6. [Nk2, Theorem 5] *Let μ be a nilpotent Lie algebra law. For the nilpotent Lie algebra $\mathfrak{n} = (\mathbb{R}^n, \mu)$ with a pre-Einstein derivation ϕ , the following conditions are equivalent:*

- (i) \mathfrak{n} is an Einstein nilradical.
- (ii) The orbit $G_{\phi} \cdot \mu$ is closed in V .
- (iii) The function $\rho_{\mu} : G_{\phi} \rightarrow \mathbb{R}$ defined by $\rho_{\mu}(g) = \|g \cdot \mu\|^2$ attains the minimum.

Suppose the orbit $G_{\phi} \cdot \mu$ is not closed. Then there exists a unique closed orbit $G_{\phi} \cdot \mu_0 \subset \overline{G_{\phi} \cdot \mu}$ such that the algebra $\mathfrak{n}_0 = (\mathbb{R}^n, \mu_0)$ is an Einstein nilradical not isomorphic to \mathfrak{n} and there exists a symmetric matrix $A \in \mathfrak{g}_{\phi}$, with integer eigenvalues, such that $\lim_{t \rightarrow \infty} \exp(tA) \cdot \mu = \mu_0$.

The inner product in V (and so the norm used in Theorem 2.6) is the induced by the canonical inner product of \mathbb{R}^n (see [L3, Equation (3.3)]).

Definition 2.7. Let $\{X_1, \dots, X_n\}$ be a basis for a nilpotent Lie algebra \mathfrak{n} , with $[X_i, X_j] = \sum_{k=1}^n c_{ij}^k X_k$. The basis $\{X_i\}$ is called *nice*, if for all i, j , $\#\{k : c_{ij}^k \neq 0\} \leq 1$ and for all i, k , $\#\{j : c_{ij}^k \neq 0\} \leq 1$.

In dimension 6, there is only one real nilpotent Lie algebra that does not admit a nice basis, it is denoted by $L_{6,11}$ in [Gr]. In fact, following Graaf's classification, $L_{6,11}$ is the only which is not written in a nice basis. The dimensions of the descending central series and derived series for $L_{6,11}$ are $(6, 3, 2, 1, 0)$ and $(6, 3, 0)$, respectively, and it is easy to see that if a 6-dimensional nilpotent Lie algebra with the same dimensions of the descending central series and derived series that $L_{6,11}$ admit a nice basis, then any such algebra must be isomorphic to $L_{6,12}$, $L_{6,13}$, $L_{5,6} \oplus \mathbb{R}x_6$ or $L_{5,7} \oplus \mathbb{R}x_6$. Thus $L_{6,11}$ can not admit a nice basis. In dimension 7, any

complex nilpotent Lie algebra of rank greater or equal than 3 has a nice basis (by a straightforward verification of the list [M]). It is proved in [Nk1] that every filiform Lie algebra admitting an \mathbb{N} -gradation have a nice basis. Although the nice basis condition looks “very exclusive”, it is satisfied by many families of nilpotent Lie algebras.

Let $\{E_{ij}\}$ be the canonical basis of $\mathfrak{gl}_n(\mathbb{R})$ and let $\alpha_{ij}^k = E_{kk} - E_{ii} - E_{jj}$ with $1 \leq i < j \leq n$ and $1 \leq k \leq n$ denote the weights of the representation of $\mathfrak{gl}_n(\mathbb{R})$ on V (via the action of $\mathrm{GL}_n(\mathbb{R})$ on V). Let us now state Nikolayevsky’s nice basis criterium.

Theorem 2.8. [Nk2, Theorem 3.] *Let $\mathfrak{n} = (\mathbb{R}^n, \mu)$ be a nilpotent Lie algebra, with $\mu = \sum c_{ij}^k \mu_{ij}^k$, $\mu \neq 0$. Let F be an ordered set of weights that are related with μ (i.e. $\alpha_{ij}^k \in F$ if and only if $c_{ij}^k \neq 0$), set $m = \#F$ and define the (Gram) matrix $U \in \mathrm{M}(m, \mathbb{R})$ as*

$$U_{p,q} := \mathrm{tr}(F(p)F(q)).$$

If the canonical basis $\{e_i\}_{i=1}^n$ is a nice basis then the following conditions are equivalent:

- (i) \mathfrak{n} is an Einstein nilradical.
- (ii) The vector of minimum norm in the convex hull of F is in the interior of the hull.
- (iii) The equation $Ux = [1]_m$ has at least one solution x with positive coordinates.

3. CLASSIFICATION

The purpose of this section is to describe the classification and to present four cases which illustrate how we get the full classification of 7-dimensional Einstein nilradical. In [F1], each nilpotent Lie algebra is studied in detail.

Undoubtedly, our principal tool is Nikolayevsky’s nice basis criterium and Carles’ classification. Following [M], the only algebras of rank ≥ 1 that are not written in a nice basis are 1.2(ii), 1.2(iv), 1.3(i_λ), 1.3(ii), 1.3(v), 1.11, 1.12, 1.13, 1.14, 1.15, 1.16, 1.17, 1.18, 1.21, 2.2, 2.11, 2.24, 2.25, 2.26, 2.27, 2.37. Some of these algebras may admit a nice basis or not. Recall that a rank-zero nilpotent Lie algebra can not be an Einstein nilradical as it does not admit an \mathbb{N} -gradation.

By using that any nilpotent Lie algebra of dimension less or equal than 6 is an Einstein nilradical ([L2], [W]), one obtains that any decomposable 7-dimensional nilpotent Lie algebra is an Einstein nilradical.

Theorem 3.1. *The classification of 7-dimensional indecomposable Einstein nilradicals is given according to their rank in Tables 1, 2, 3 and 4.*

The notation in the tables is as follows:

✓ := Yes, - := No, EN := Einstein Nilradical, Min := Minimum of $F|_{\mathrm{GL}_n(\mathbb{R}) \cdot \mu}$, dim DCS := Dimension of descending central series and dim Der := Dimension of the algebra of derivations.

In dim DCS we omit the first term, which is always 7. So for example the algebra $\mathfrak{n} = (\mathbb{R}^n, \mu)$ with $\mu = 1.3(i_0)$ is not an Einstein nilradical and hence minimum of $F|_{\text{GL}_n(\mathbb{R}) \cdot \mu}$ does not exist, the dimension of its algebra of derivations is 13 and dimension of descending central series is $(7, 4, 2, 1, 0)$ that correspond to $\mathfrak{n} = \mathfrak{n}_0 \geq \mathfrak{n}_1 = [\mathfrak{n}, \mathfrak{n}_0] \geq \mathfrak{n}_2 = [\mathfrak{n}, \mathfrak{n}_1] \geq \mathfrak{n}_3 = [\mathfrak{n}, \mathfrak{n}_2] \geq \mathfrak{n}_4 = [\mathfrak{n}, \mathfrak{n}_3] = 0$.

Example 3.2. Exhibiting a nilsoliton inner product

$\mathfrak{g}_{1.17}$

$$\mu := \begin{cases} [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_6, [e_1, e_6] = e_7, [e_2, e_3] = e_5, \\ [e_2, e_5] = e_6, [e_2, e_6] = e_7, [e_3, e_4] = -e_7, [e_3, e_5] = e_7 \end{cases}$$

This algebra has rank 1 and its maximal torus of derivations is generated by $\text{Diag}(1, 1, 2, 3, 3, 4, 5)$. It follows from Corollary 2.3 that if $\mathfrak{g}_{1.17}$ is an Einstein nilradical, then there must be a Lie algebra law $\tilde{\mu}$ in $\text{GL}_7(\mathbb{R}) \cdot \mu$ with $\text{Diag}(1, 1, 2, 3, 3, 4, 5)$ as Einstein derivation and moment map equal to

$$(3.1) \quad \mathfrak{m}(\tilde{\mu}) = \text{Diag} \left(-\frac{23}{47}, -\frac{23}{47}, -\frac{27}{94}, -\frac{4}{47}, -\frac{4}{47}, \frac{11}{94}, \frac{15}{47} \right)$$

However, any algebra law admitting $\text{Diag}(1, 1, 2, 3, 3, 4, 5)$ as a derivation is of the form

$$\mu(a_1, \dots, a_{13}) := \begin{cases} [e_1, e_2] = a_1 e_3, [e_1, e_3] = a_2 e_4 + a_3 e_5, \\ [e_1, e_4] = a_4 e_6, [e_1, e_5] = a_5 e_6, [e_1, e_6] = a_6 e_7, \\ [e_2, e_3] = a_7 e_4 + a_8 e_5, [e_2, e_4] = a_9 e_6, [e_2, e_5] = a_{10} e_6, \\ [e_2, e_6] = a_{11} e_7, [e_3, e_4] = a_{12} e_7, [e_3, e_5] = a_{13} e_7. \end{cases}$$

If J represents the Jacobi condition, then Einstein nilradicals of eigenvalue type $(1 < 2 < 3 < 4 < 5; 2, 1, 2, 1, 1)$ are characterized by $J(\mu(a_1, \dots, a_{13})) = 0$ and $\mathfrak{m}(\mu(a_1, \dots, a_{13}))$ as in (3.1), or equivalently, by the following polynomial equations system:

$$\begin{aligned} J(\mu(a_1, \dots, a_{13})) & \begin{cases} -a_{10}a_6 + a_5a_{11} + a_1a_{13} = 0, \\ -a_9a_6 + a_4a_{11} + a_1a_{12} = 0, \\ -a_8a_5 + a_3a_{10} - a_7a_4 + a_2a_9 = 0, \\ -2a_1^2 - 2a_2^2 - 2a_3^2 - 2a_4^2 - 2a_5^2 - 2a_6^2 = -\frac{23}{47}, \\ -2a_2a_7 - 2a_3a_8 - 2a_4a_9 - 2a_5a_{10} - 2a_6a_{11} = 0, \\ -2a_1^2 - 2a_7^2 - 2a_8^2 - 2a_9^2 - 2a_{10}^2 - 2a_{11}^2 = -\frac{23}{47}, \\ 2a_1^2 - 2a_2^2 - 2a_3^2 - 2a_7^2 - 2a_8^2 - 2a_{12}^2 - 2a_{13}^2 = -\frac{27}{94}, \\ 2a_2^2 - 2a_4^2 + 2a_7^2 - 2a_9^2 - 2a_{12}^2 = -\frac{4}{47}, \\ 2a_2a_3 - 2a_4a_5 + 2a_7a_8 - 2a_9a_{10} - 2a_{12}a_{13} = 0, \\ 2a_3^2 - 2a_5^2 + 2a_8^2 - 2a_{10}^2 - 2a_{13}^2 = -\frac{4}{47}, \\ 2a_4^2 + 2a_5^2 - 2a_6^2 + 2a_9^2 + 2a_{10}^2 - 2a_{11}^2 = \frac{11}{94}, \\ 2a_6^2 + 2a_{11}^2 + 2a_{12}^2 + 2a_{13}^2 = \frac{15}{47}. \end{cases} \\ \mathfrak{m}(\mu(a_1, \dots, a_{13})) & \end{aligned}$$

Since an eigenvalue type may have many non isomorphic Einstein nilradicals, we must find a solution such that its nilpotent Lie algebra law is isomorphic to $\mathfrak{g}_{1.17}$. A priori, we have found a pre-Einstein derivation

to all 7-dimensional indecomposable algebras in [M] and $\mathfrak{g}_{1.17}$ is the only indecomposable nilpotent Lie algebra that has a pre-Einstein derivation of eigenvalue type $(1 < 2 < 3 < 4 < 5; 2, 1, 2, 1, 1)$, thus we only have to verify the indecomposability property between solutions of such equations system to show that $\mathfrak{g}_{1.17}$ is an Einstein nilradical. By using Gröbner basis to solve the polynomial equations system we find some solutions given by:

$$(3.2) \quad \begin{aligned} a_1 &= \pm \frac{\sqrt{611}}{94}, a_2 = 0, a_3 = \pm \frac{\sqrt{235}}{47}, a_4 = 0, a_5 = \pm \frac{\sqrt{611}}{94}, a_6 = 0, \\ a_7 &= \pm \frac{\sqrt{235}}{94}, a_8 = 0, a_9 = \pm \frac{\sqrt{611}}{94}, a_{10} = 0, a_{11} = \pm \frac{\sqrt{705}}{94}, \\ a_{12} &= 0, a_{13} = -\frac{a_5 a_{11}}{a_1}. \end{aligned}$$

By fixing a solution we have an Einstein nilradical $(\mathbb{R}^n, \tilde{\mu})$ given by

$$\tilde{\mu} := \begin{cases} [e_1, e_2] = \frac{\sqrt{611}}{94} e_3, [e_1, e_3] = \frac{\sqrt{235}}{47} e_5, [e_1, e_5] = \frac{\sqrt{611}}{94} e_6, [e_2, e_3] = \frac{\sqrt{235}}{94} e_4, \\ [e_2, e_4] = \frac{\sqrt{611}}{94} e_6, [e_2, e_6] = \frac{\sqrt{705}}{94} e_7, [e_3, e_5] = -\frac{\sqrt{705}}{94} e_7 \end{cases}$$

As $(\mathbb{R}^n, \tilde{\mu})$ is indecomposable by the analysis above, this must be isomorphic to (\mathbb{R}^n, μ) . To find an isomorphism, since the map $\text{Diag}(1, 1, 2, 3, 3, 4, 5)$ with respect to the basis $\{e_i\}$ is a derivation of both algebras, we can assume that an isomorphism is given by a matrix in $\text{GL}_7(\mathbb{R})$ that commutes with $\text{Diag}(1, 1, 2, 3, 3, 4, 5)$,

$$g = \text{Diag} \left(\begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}, b_{3,3}, \begin{pmatrix} b_{4,4} & b_{4,5} \\ b_{5,4} & b_{5,5} \end{pmatrix}, b_{6,6}, b_{7,7} \right)$$

By solving the equation $g \cdot \mu = \tilde{\mu}$ we get

$$g = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{1222}}{47} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{65}}{47} & \frac{\sqrt{65}}{47} & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{130}}{47} & -\frac{\sqrt{130}}{47} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{13\sqrt{470}}{2209} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{65\sqrt{3}}{2209} \end{pmatrix},$$

and thus $\mathfrak{g}_{1.17}$ is an Einstein nilradical, as was to be shown.

Rank one					
n	EN	pre-Einstein derivation	Min	dim Der	dim DCS
1.01(i)	-	(0, 1, 0, 1, 1, 1, 1)	-	11	(4, 3, 2, 1)
1.01(ii)	-	(0, 1, 0, 1, 1, 1, 1)	-	12	(4, 3, 2, 1)
1.02	-	$\frac{1}{2}(0, 1, 1, 1, 2, 2, 3)$	-	11	(5, 4, 2, 1)
1.03	-	$\frac{2}{3}(0, 1, 1, 1, 1, 2, 2)$	-	11	(5, 4, 2, 1)

Rank one					
n	EN	pre-Einstein derivation	Min	dim Der	dim DCS
$1.1(i_\lambda)$ $\lambda \neq 0, 1$	✓	$\frac{1}{5}(1, 2, 3, 4, 5, 6, 7)$	0.714	10	(5, 4, 3, 2, 1)
$1.1(i_\lambda)$ $\lambda = 0$	-	$\frac{1}{5}(1, 2, 3, 4, 5, 6, 7)$	-	10	(5, 4, 3, 2, 1)
$1.1(i_\lambda)$ $\lambda = 1$	-	$\frac{1}{5}(1, 2, 3, 4, 5, 6, 7)$	-	11	(5, 4, 3, 2, 1)
$1.1(ii)$	-	$\frac{1}{5}(1, 2, 3, 4, 5, 6, 7)$	-	11	(5, 4, 3, 2, 1)
$1.1(iii)$	✓	$\frac{1}{5}(1, 2, 3, 4, 5, 6, 7)$	0.714	10	(5, 4, 3, 2)
$1.1(iv)$	-	$\frac{1}{5}(1, 2, 3, 4, 5, 6, 7)$	-	11	(5, 4, 2, 1)
$1.1(v)$	-	$\frac{1}{5}(1, 2, 3, 4, 5, 6, 7)$	-	10	(4, 3, 2, 1)
$1.1(vi)$	-	$\frac{1}{5}(1, 2, 3, 4, 5, 6, 7)$	-	11	(4, 3, 2, 1)
$1.2(i_\lambda)$ $\lambda \neq 0, 1$	✓	$\frac{4}{11}(1, 1, 2, 2, 3, 3, 4)$	0.846	12	(4, 3, 1)
$1.2(i_\lambda)$ $\lambda = 0$	✓	$\frac{4}{11}(1, 1, 2, 2, 3, 3, 4)$	0.846	12	(4, 3, 1)
$1.2(i_\lambda)$ $\lambda = 1$	✓	$\frac{4}{11}(1, 1, 2, 2, 3, 3, 4)$	0.846	12	(4, 3, 1)
$1.2(ii)$	-	$\frac{4}{11}(1, 1, 2, 2, 3, 3, 4)$	-	12	(4, 3, 1)
$1.2(iii)$	-	$\frac{4}{11}(1, 1, 2, 2, 3, 3, 4)$	-	12	(4, 3, 1)
$1.2(iv)$	-	$\frac{4}{11}(1, 1, 2, 2, 3, 3, 4)$	-	12	(4, 2, 1)
$1.3(i_\lambda)$ $\lambda \neq 0$	✓	$\frac{5}{17}(1, 2, 2, 3, 3, 4, 5)$	0.895	13	(4, 2, 1)
$1.3(i_\lambda)$ $\lambda = 0$	-	$\frac{5}{17}(1, 2, 2, 3, 3, 4, 5)$	-	13	(4, 2, 1)
$1.3(ii)$	-	$\frac{5}{17}(1, 2, 2, 3, 3, 4, 5)$	-	14	(4, 2, 1)
$1.3(iii)$	✓	$\frac{5}{17}(1, 2, 2, 3, 3, 4, 5)$	0.895	13	(4, 2, 1)
$1.3(iv)$	-	$\frac{5}{17}(1, 2, 2, 3, 3, 4, 5)$	-	13	(4, 2)
$1.3(v)$	-	$\frac{5}{17}(1, 2, 2, 3, 3, 4, 5)$	-	13	(3, 2, 1)
1.4	✓	$\frac{17}{100}(1, 3, 4, 5, 6, 7, 8)$	0.820	12	(5, 4, 3, 2, 1)
1.5	✓	$\frac{5}{31}(1, 3, 4, 5, 6, 7, 9)$	0.738	11	(5, 4, 3, 2)
1.6	✓	$\frac{5}{34}(1, 4, 5, 6, 7, 8, 9)$	0.895	12	(5, 4, 3, 2, 1)
1.7	✓	$\frac{5}{29}(2, 3, 4, 5, 6, 7, 8)$	1.04	15	(4, 2)
1.8	-	$\frac{20}{139}(2, 4, 3, 6, 7, 8, 10)$	-	11	(4, 2, 1)
1.9	-	$\frac{10}{67}(2, 3, 6, 5, 7, 8, 9)$	-	14	(4, 3, 1)
1.10	✓	$\frac{45}{353}(2, 3, 5, 7, 8, 9, 11)$	0.792	11	(5, 4, 2, 1)
1.11	✓	$\frac{6}{25}(1, 2, 3, 3, 4, 5, 6)$	0.806	11	(4, 3, 2, 1)
1.12	✓	$\frac{25}{107}(1, 2, 4, 3, 4, 5, 6)$	0.863	12	(4, 3, 2, 1)
1.13	✓	$\frac{13}{58}(1, 2, 3, 4, 5, 5, 6)$	0.853	12	(5, 4, 2, 1)
1.14	✓	$\frac{9}{43}(1, 2, 3, 4, 5, 5, 7)$	0.741	11	(5, 4, 2, 1)

Rank one					
n	EN	pre-Einstein derivation	Min	dim Der	dim DCS
1.15	✓	$\frac{15}{76}(1, 3, 4, 4, 5, 6, 7)$	0.927	13	(4, 3, 2, 1)
1.16	✓	$\frac{11}{40}(1, 2, 3, 3, 4, 4, 5)$	1.05	15	(4, 2, 1)
1.17	✓	$\frac{19}{65}(1, 1, 2, 3, 3, 4, 5)$	0.692	11	(5, 4, 2, 1)
1.18	✓	$\frac{23}{89}(1, 2, 3, 3, 4, 5, 5)$	0.947	13	(4, 3, 1)
1.19	✓	$\frac{13}{29}(1, 1, 1, 2, 2, 3, 3)$	0.853	11	(4, 2)
1.20	-	$\frac{8}{47}(1, 2, 3, 5, 6, 7, 8)$	-	11	(4, 3, 2, 1)
1.21	-	$\frac{25}{113}(1, 2, 3, 3, 4, 5, 7)$	-	11	(4, 3, 2, 1)

TABLE 1. Classification of 7-dimensional indecomposable Einstein nilradicals. Rank one case.

Example 3.3. Showing that the G_ϕ -orbit is closed

$\mathfrak{g}_{1.3(i_\lambda)}$ with $\lambda \neq 0$

$$\mu := \begin{cases} [e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_1, e_4] = e_6, [e_1, e_6] = e_7, [e_2, e_3] = e_6, \\ [e_2, e_4] = \lambda e_7, [e_2, e_5] = e_7, [e_3, e_5] = e_7 \end{cases}$$

For any $\lambda \neq 0$, $\mathfrak{g}_{1.3(i_\lambda)}$ is an Einstein nilradical. We prove this by contradiction; assume that $\mathfrak{g}_{1.3(i_\lambda)}$ is not an Einstein nilradical. The derivation ϕ given by the diagonal matrix $\frac{5}{17} \text{Diag}(1, 2, 2, 3, 3, 4, 5)$ with respect to the basis $\{e_i\}$ is pre-Einstein. It follows from Theorem 2.6 that the orbit $G_\phi \cdot \mu$ is not closed and so there exists $Y \in \mathfrak{g}_\phi$, Y is a symmetric matrix, such that μ degenerates by the action of the one-parameter subgroup $\exp(tY)$ as $t \rightarrow \infty$. As $Y \in \mathfrak{g}_\phi$ and Y is symmetric, then there exists $X = \text{Diag}(a_1, \dots, a_7) \in \mathfrak{g}_\phi$ and $A(\alpha)$, $B(\beta)$ in $\text{SO}_2(\mathbb{R})$ such that

$$Y = \text{Diag}(1, A(-\alpha), B(-\beta), 1, 1) X \text{Diag}(1, A(\alpha), B(\beta), 1, 1).$$

As the action is continuous, it follows that μ also degenerates by the action of

$$g_t := \exp(tX) \text{Diag}(1, A(\alpha), B(\beta), 1, 1)$$

as $t \rightarrow \infty$. The contradiction will be found in this last fact. The action of g_t in μ is

$$g_t \cdot \mu = \begin{cases} [e_1, e_2] = e^{-t(a_1+a_2-a_4)} \cos(\alpha - \beta) e_4 - e^{-t(a_1+a_2-a_5)} \sin(\alpha - \beta) e_5, \\ [e_1, e_3] = e^{-t(a_1+a_3-a_4)} \sin(\alpha - \beta) e_4 + e^{-t(a_1+a_3-a_5)} \cos(\alpha - \beta) e_5, \\ [e_1, e_4] = e^{-t(a_1+a_4-a_6)} \cos(\beta) e_6, [e_1, e_5] = e^{-t(a_1+a_5-a_6)} \sin(\beta) e_6, \\ [e_1, e_6] = e^{-t(a_1+a_6-a_7)} e_7, [e_2, e_3] = e^{-t(a_2+a_3-a_6)} e_6, \\ [e_2, e_4] = e^{-t(a_2+a_4-a_7)} f_{2,4}(\alpha, \beta) e_7, \\ [e_2, e_5] = e^{-t(a_2+a_5-a_7)} f_{2,5}(\alpha, \beta) e_7, \\ [e_3, e_4] = e^{-t(a_3+a_4-a_7)} f_{3,4}(\alpha, \beta) e_7, \\ [e_3, e_5] = e^{-t(a_3+a_5-a_7)} f_{3,5}(\alpha, \beta) e_7. \end{cases}$$

with

$$\begin{aligned} f_{2,4}(\alpha, \beta) &= (\cos(\alpha) \cos(\beta) \lambda + \sin(\alpha) \sin(\beta) - \cos(\alpha) \sin(\beta)), \\ f_{2,5}(\alpha, \beta) &= (\cos(\alpha) \sin(\beta) \lambda - \sin(\alpha) \cos(\beta) + \cos(\alpha) \cos(\beta)), \\ f_{3,4}(\alpha, \beta) &= (\sin(\alpha) \cos(\beta) \lambda - \cos(\alpha) \sin(\beta) - \sin(\alpha) \sin(\beta)), \\ f_{3,5}(\alpha, \beta) &= (\sin(\alpha) \sin(\beta) \lambda + \cos(\alpha) \cos(\beta) + \sin(\alpha) \cos(\beta)). \end{aligned}$$

According to values of α and β some terms are zero in the Lie algebra law $g_t \cdot \mu$ and as the degeneration is determined by non-zero terms, our attention is in the exponent of the exponential factor of such terms; when $t > 0$, such exponent must be non negative.

It is easy to see that pairs of functions $\{f_{2,4}, f_{2,5}\}$, $\{f_{2,4}, f_{3,4}\}$, $\{f_{2,5}, f_{3,5}\}$ and $\{f_{3,4}, f_{3,5}\}$ do not vanish simultaneously (as sin and cos). We have the following cases depending on which terms are non zero.

I) $\cos(\beta)$ and $\sin(\beta)$ are non zero

1. $\cos(\alpha - \beta)$, $f_{2,4}$, $f_{3,4}$ are non zero

If this is the case then it must be that a_1, \dots, a_7 satisfy: $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 = 0$, $a_1 + 2a_2 + 2a_3 + 3a_4 + 3a_5 + 4a_6 + 5a_7 = 0$ since $X \in \mathfrak{g}_\phi$ and $a_1 + a_6 - a_7 \geq 0$, $a_2 + a_3 - a_6 \geq 0$, $a_1 + a_4 - a_6 \geq 0$, $a_1 + a_5 - a_6 \geq 0$, $a_1 + a_2 - a_4 \geq 0$, $a_1 + a_3 - a_5 \geq 0$, $a_2 + a_4 - a_7 \geq 0$ and $a_3 + a_4 - a_7 \geq 0$.

Instead of solving the inequalities system, we can do the next trick: we introduce a new variable c_i for each inequality q_i and we have $q_i - c_i^2 = 0$ and so we must solve the polynomial equations system:

$$\begin{cases} a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 = 0, \\ a_1 + 2a_2 + 2a_3 + 3a_4 + 3a_5 + 4a_6 + 5a_7 = 0, \\ a_1 + a_6 - a_7 - c_1^2 = 0, a_2 + a_3 - a_6 - c_2^2 = 0, \\ a_1 + a_4 - a_6 - c_3^2 = 0, a_1 + a_5 - a_6 - c_4^2 = 0, \\ a_1 + a_2 - a_4 - c_5^2 = 0, a_1 + a_3 - a_5 - c_6^2 = 0, \\ a_2 + a_4 - a_7 - c_7^2 = 0, a_3 + a_4 - a_7 - c_8^2 = 0. \end{cases}$$

We set that $c_1^2 + 4c_4^2 + 5c_5^2 + 2c_6^2 + 2c_7^2 + 5c_8^2 = 0$, $c_2^2 - 6c_5^2 - 6c_8^2 - 4c_4^2 - 2c_6^2 - 2c_7^2 = 0$ and $c_3^2 + c_5^2 + c_8^2 - c_4^2 - c_6^2 - c_7^2 = 0$. By the first equality c_1, c_4, c_5, c_6, c_7 and c_8 are zero and so c_2 and c_3 are zero too. The degeneration is therefore trivial and so in this case we get a contradiction. Remaining cases are similar and we can see them in [F1]. So, no matter which case is, the degeneration is trivial. Hence $\mathfrak{g}_{1.3(i_\lambda)}$, with $\lambda \neq 0$, must be an Einstein Nilradical.

Rank two					
n	EN	pre-Einstein derivation	Min	dim Der	dim DCS
$2.1(i_\lambda)$ $\lambda \neq 0, 1$	✓	$\frac{2}{19}(3, 5, 6, 8, 9, 11, 14)$	0.905	14	(4, 2, 1)

Rank two					
n	EN	pre-Einstein derivation	Min	dim Der	dim DCS
2.1(i_λ) $\lambda = 0$	-	$\frac{2}{19}(3, 5, 6, 8, 9, 11, 14)$	-	14	(4, 2, 1)
2.1(i_λ) $\lambda = 1$	✓	$\frac{2}{19}(3, 5, 6, 8, 9, 11, 14)$	0.905	14	(4, 2, 1)
2.1(ii)	✓	$\frac{2}{19}(3, 5, 6, 8, 9, 11, 14)$	0.905	14	(4, 2, 1)
2.1(iii)	✓	$\frac{2}{19}(3, 5, 6, 8, 9, 11, 14)$	0.905	14	(3, 2, 1)
2.1(iv)	-	$\frac{2}{19}(3, 5, 6, 8, 9, 11, 14)$	-	14	(3, 1)
2.1(v)	-	$\frac{2}{19}(3, 5, 6, 8, 9, 11, 14)$	-	14	(4, 2)
2.2	-	$\frac{1}{2}(1, 1, 1, 2, 2, 2, 3)$	-	15	(4, 1)
2.3	✓	$\frac{2}{37}(1, 16, 17, 18, 19, 20, 21)$	1.06	13	(5, 4, 3, 2, 1)
2.4	✓	$\frac{7}{52}(1, 4, 5, 6, 7, 8, 11)$	0.743	12	(5, 4, 3, 2)
2.5	✓	$\frac{1}{5}(1, 2, 3, 4, 5, 6, 7)$	0.714	12	(5, 4, 2, 1)
2.6	✓	$\frac{1}{52}(10, 23, 33, 43, 56, 53, 76)$	0.743	12	(5, 4, 2, 1)
2.7	✓	$\frac{1}{18}(3, 10, 13, 16, 23, 19, 22)$	0.9	13	(5, 4, 2, 1)
2.8	✓	$\frac{1}{12}(3, 5, 8, 11, 13, 14, 16)$	0.857	13	(5, 4, 2)
2.9	✓	$\frac{9}{28}(1, 1, 2, 3, 3, 4, 4)$	0.824	12	(5, 4, 2)
2.10	-	$\frac{1}{5}(1, 2, 6, 3, 4, 5, 7)$	-	12	(4, 3, 2, 1)
2.11	✓	$\frac{1}{36}(9, 19, 28, 28, 37, 47, 46)$	0.947	14	(4, 3, 1)
2.12	✓	$\frac{1}{9}(3, 5, 5, 8, 8, 11, 13)$	0.9	14	(4, 2)
2.13	✓	$\frac{1}{60}(16, 21, 48, 37, 53, 69, 90)$	0.698	12	(4, 3, 2, 1)
2.14	✓	$\frac{1}{5}(1, 3, 2, 4, 5, 6, 7)$	0.714	12	(4, 3, 2, 1)
2.15	✓	$\frac{1}{5}(1, 3, 3, 4, 5, 6, 7)$	0.833	13	(4, 3, 2, 1)
2.16	✓	$\frac{1}{27}(5, 17, 20, 22, 27, 32, 37)$	0.931	14	(4, 3, 2, 1)
2.17	✓	$\frac{1}{12}(4, 5, 8, 9, 13, 14, 17)$	0.857	13	(4, 3, 1)
2.18	✓	$\frac{1}{68}(20, 31, 60, 51, 71, 82, 91)$	0.971	15	(4, 3, 1)
2.19	-	$\frac{1}{4}(1, 2, 4, 3, 4, 5, 5)$	-	15	(4, 3, 1)
2.20	✓	$\frac{2}{37}(5, 16, 10, 21, 15, 20, 25)$	1.06	16	(4, 2, 1)
2.21	✓	$\frac{1}{31}(8, 19, 24, 27, 32, 35, 43)$	1.07	16	(4, 2, 1)
2.22	✓	$\frac{1}{19}(5, 14, 10, 15, 24, 20, 25)$	0.950	14	(4, 2, 1)
2.23	-	$\frac{4}{11}(1, 1, 2, 2, 3, 3, 4)$	-	13	(3, 1)
2.24	✓	$\frac{1}{17}(5, 9, 10, 14, 19, 19, 24)$	0.895	13	(4, 2, 1)
2.25	✓	$\frac{5}{17}(1, 2, 2, 3, 3, 4, 5)$	0.895	14	(3, 2, 1)
2.26	✓	$\frac{1}{17}(7, 10, 7, 17, 14, 21, 24)$	0.895	13	(4, 2)
2.27	✓	$\frac{1}{15}(6, 7, 14, 13, 13, 19, 20)$	1.15	17	(3, 2)
2.28	✓	$\frac{4}{37}(4, 5, 6, 8, 9, 10, 14)$	1.06	16	(3, 1)
2.29	-	$\frac{1}{41}(15, 22, 30, 29, 37, 52, 59)$	-	14	(3, 2)
2.30	✓	$\frac{4}{27}(2, 4, 5, 5, 6, 8, 10)$	0.931	15	(3, 2, 1)
2.31	✓	$\frac{1}{39}(14, 15, 27, 29, 42, 43, 57)$	0.848	13	(4, 2, 1)
2.32	✓	$\frac{2}{27}(3, 10, 8, 13, 11, 16, 19)$	0.931	14	(4, 2, 1)

Rank two					
n	EN	pre-Einstein derivation	Min	dim Der	dim DCS
2.33	✓	$\frac{1}{33}(10, 18, 15, 28, 33, 38, 48)$	0.805	12	(4, 2, 1)
2.34	✓	$\frac{1}{47}(22, 20, 21, 42, 43, 62, 64)$	0.854	12	(4, 2)
2.35	✓	$\frac{1}{13}(5, 6, 7, 11, 12, 17, 18)$	0.867	12	(4, 2)
2.36	✓	$\frac{1}{23}(18, 13, 10, 15, 28, 23, 33)$	1.10	16	(3, 1)
2.37	✓	$\frac{4}{11}(1, 1, 2, 2, 3, 3, 4)$	0.846	13	(4, 3, 1)
2.38	✓	$\frac{7}{16}(1, 1, 2, 2, 2, 3, 3)$	1.14	16	(3, 2)
2.39	✓	$\frac{1}{16}(5, 11, 10, 16, 15, 21, 20)$	1.14	17	(4, 2)
2.40	✓	$\frac{1}{23}(9, 10, 19, 18, 28, 29, 27)$	1.10	16	(4, 2)
2.41	✓	$\frac{1}{7}(2, 3, 5, 6, 7, 8, 10)$	0.875	13	(4, 3, 1)
2.42	-	$\frac{1}{41}(11, 22, 30, 33, 41, 52, 55)$	-	14	(4, 2)
2.43	✓	$\frac{1}{37}(11, 29, 20, 40, 31, 42, 51)$	1.06	16	(4, 2)
2.44	✓	$\frac{1}{37}(15, 19, 23, 34, 38, 42, 53)$	1.06	16	(4, 1)
2.45	✓	$\frac{1}{14}(6, 7, 11, 12, 13, 19, 18)$	1.17	17	(3, 1)

TABLE 2. Classification of 7-dimensional indecomposable Einstein nilradicals. Rank two case.

Example 3.4. Degeneration by action of the group G_ϕ

$\mathfrak{g}_{2.2}$

$$\mu := \begin{cases} [e_1, e_2] = e_5, [e_1, e_3] = e_6, [e_1, e_4] = 2e_7, [e_2, e_3] = e_4, [e_2, e_6] = e_7, \\ [e_3, e_5] = -e_7, [e_3, e_6] = e_7 \end{cases}$$

$\mathfrak{g}_{2.2}$ is a nilpotent Lie algebra of rank 2 with a maximal torus of derivations generated by $D_1 = \text{Diag}(1, 0, 0, 0, 1, 1, 1)$ and $D_2 = \text{Diag}(0, 1, 1, 2, 1, 1, 2)$ (with respect to the basis $\{e_i\}$). If $\phi = aD_1 + bD_2$ is a pre-Einstein derivation, then a, b are found by solving the linear equations system

$$\begin{cases} \text{tr}(\phi D_1) = \text{tr}(D_1), \\ \text{tr}(\phi D_2) = \text{tr}(D_2). \end{cases}$$

We get that $\phi = \frac{1}{2}D_1 + \frac{1}{2}D_2 = \frac{1}{2}(1, 1, 1, 2, 2, 2, 3)$, and so if $\mathfrak{g}_{2.2}$ is an Einstein nilradical, then it has eigenvalue type $(1 < 2 < 3; 3, 3, 1)$. The method used in Example 3.2 applied to this algebra is very cumbersome because the polynomial equations system attached to the type $(1 < 2 < 3; 3, 3, 1)$ has infinitely many solutions whose algebras are pairwise non isomorphic; the curve $3.1(\lambda)$ with $\lambda \neq 0 \neq 1$ is Einstein nilradical of type $(1 < 2 < 3; 3, 3, 1)$. By analyzing solutions of such system, we see that there is no any solution that corresponds to $\mathfrak{g}_{2.2}$. Therefore we use another way to proof that $\mathfrak{g}_{2.2}$ is not an Einstein Nilradical. By Theorem 2.6, we try to find a non-trivial degeneration of $\mathfrak{g}_{2.2}$ by the action of a 1-parameter subgroup of G_ϕ . Let $X \in \mathfrak{g}_\phi$ be a diagonal matrix, $X = \text{Diag}(a_1, \dots, a_7)$. As $\text{tr}(X\phi) = 0$ and $\text{tr}(X) = 0$ then $a_4 = -a_5 - a_6 - 2a_7$ and $a_1 = a_7 - a_2 - a_3$. The action of

$g_t = \exp(tX)$ on μ is

$$\mu_t := \begin{cases} [e_1, e_2] = e^{t(-a_7+a_3+a_5)}e_5, [e_1, e_3] = e^{t(-a_7+a_2+a_6)}e_6, \\ [e_1, e_4] = 2e^{t(a_2+a_3+a_5+a_6+2a_7)}e_7, [e_2, e_3] = e^{t(-a_2-a_3-a_5-a_6-2a_7)}e_4, \\ [e_2, e_6] = e^{t(a_7-a_2-a_6)}e_7, [e_3, e_5] = -e^{t(a_7-a_3-a_5)}e_7, \\ [e_3, e_6] = e^{t(a_7-a_3-a_6)}e_7 \end{cases}$$

To find a non-trivial degeneration as $t \rightarrow \infty$, we need that exponents be negative. By doing the same trick as in Example 3.3, we get a polynomial equations system

$$\begin{cases} -a_7 + a_3 + a_5 = -b_1^2, -a_7 + a_2 + a_6 = -b_2^2 \\ a_2 + a_3 + a_5 + a_6 + 2a_7 = -b_3^2 \\ -a_2 - a_3 - a_5 - a_6 - 2a_7 = -b_4^2 \\ a_7 - a_2 - a_6 = -b_5^2, a_7 - a_3 - a_5 = -b_6^2 \\ a_7 - a_3 - a_6 = -b_8^2 \end{cases}$$

whose solutions are given by

$$(3.3) \quad \begin{aligned} a_2 &= \frac{1}{4}b_4^2 - a_5 - b_7^2 + \frac{3}{4}b_5^2 + \frac{3}{4}b_6^2, a_3 = -a_5 + \frac{1}{4}b_4^2 - \frac{1}{4}b_5^2 + \frac{3}{4}b_6^2, \\ a_5 &= a_5, a_6 = a_5 + b_7^2 - b_6^2, a_7 = \frac{1}{4}b_4^2 - \frac{1}{4}b_5^2 - \frac{1}{4}b_6^2, b_1 = \pm ib_6, \\ b_2 &= \pm ib_5, b_3 = \pm ib_4, b_4 = b_4, b_5 = b_5, b_6 = b_6, b_7 = b_7 \end{aligned}$$

As the solutions must be real, then $b_4 = b_5 = b_6 = 0$, and so that $b_1 = b_2 = b_3 = 0$. In order that the degeneration be non-trivial, we need $b_7 \neq 0$. By setting $b_7 = 1$ and $a_5 = 0$ we get $a_2 = -1$, $a_3 = 0$, $a_6 = 1$, $a_7 = 0$, $a_1 = 1$ and $a_4 = -1$, and thus

$$X = \text{Diag}(1, -1, 0, -1, 0, 1, 0)$$

As t tends to infinite $g_t \cdot \mu \rightarrow \tilde{\mu}$

$$\tilde{\mu} := \begin{cases} [e_1, e_2] = e_5, [e_1, e_3] = e_6, [e_1, e_4] = 2e_7, [e_2, e_3] = e_4, [e_2, e_6] = e_7, \\ [e_3, e_5] = -e_7 \end{cases}$$

$(\mathbb{R}^7, \tilde{\mu})$ is not isomorphic to (\mathbb{R}^7, μ) since $\dim \text{Der}(\mathbb{R}^7, \mu) = 15$ and $\dim \text{Der}(\mathbb{R}^7, \tilde{\mu}) = 17$. Therefore the G_ϕ -orbit of μ is not closed and in consequence $\mathfrak{g}_{2.2}$ is not an Einstein nilradical.

Rank three					
n	EN	pre-Einstein derivation	Min	dim Der	dim DCS
3.1(i_λ) $\lambda \neq 0, 1$	✓	$\frac{1}{2}(1, 1, 1, 2, 2, 2, 3)$	1	15	(4, 1)
3.1(i_λ) $\lambda = 0$	-	$\frac{1}{2}(1, 1, 1, 2, 2, 2, 3)$	-	15	(4, 1)
3.1(i_λ) $\lambda = 1$	-	$\frac{1}{2}(1, 1, 1, 2, 2, 2, 3)$	-	15	(4, 1)
3.1(iii)	-	$\frac{1}{2}(1, 1, 1, 2, 2, 2, 3)$	-	15	(3, 1)

Rank three					
n	EN	pre-Einstein derivation	Min	dim Der	dim DCS
3.2	✓	$\frac{2}{13}(1, 5, 6, 6, 7, 7, 8)$	1.18	17	(4, 2, 1)
3.3	✓	$\frac{1}{21}(5, 12, 15, 17, 27, 22, 27)$	0.954	15	(4, 2, 1)
3.4	✓	$\frac{1}{12}(6, 5, 5, 11, 11, 16, 16)$	0.857	13	(4, 2,)
3.5	✓	$\frac{1}{20}(10, 7, 11, 17, 21, 24, 28)$	0.909	14	(4, 2)
3.6	✓	$\frac{1}{13}(5, 9, 7, 14, 12, 16, 17)$	1.18	18	(4, 1)
3.7	-	$\frac{1}{3}(1, 2, 2, 2, 3, 4, 4)$	-	15	(3, 1)
3.8	✓	$\frac{1}{5}(2, 3, 4, 4, 5, 6, 7)$	1.25	19	(3, 1)
3.9	✓	$\frac{2}{13}(3, 3, 5, 6, 6, 8, 9)$	1.18	18	(3, 1)
3.10	✓	$\frac{1}{20}(12, 7, 11, 16, 19, 23, 30)$	0.909	15	(3, 1)
3.11	✓	$\frac{1}{13}(5, 7, 12, 10, 12, 17, 17)$	1.18	18	(3, 1)
3.12	✓	$\frac{5}{8}(1, 1, 1, 1, 2, 2, 2)$	1.33	19	(3)
3.13	✓	$\frac{1}{21}(8, 11, 15, 15, 19, 27, 30)$	0.954	16	(3, 2)
3.14	✓	$\frac{1}{13}(5, 9, 9, 10, 14, 14, 19)$	1.18	18	(3, 1)
3.15	✓	$\frac{1}{11}(6, 5, 7, 9, 11, 13, 16)$	1.10	17	(3, 1)
3.16	✓	$\frac{1}{12}(5, 8, 5, 8, 13, 13, 18)$	0.857	14	(3, 1)
3.17	✓	$\frac{5}{21}(1, 3, 3, 3, 4, 5, 6)$	0.954	16	(3, 2, 1)
3.18	✓	$\frac{2}{13}(3, 4, 5, 5, 6, 7, 10)$	1.18	19	(2, 1)
3.19	✓	$\frac{1}{8}(5, 6, 6, 5, 6, 11, 11)$	1.33	19	(2)
3.20	✓	$\frac{1}{5}(1, 4, 4, 5, 5, 6, 6)$	1.25	19	(4, 2)
3.21	✓	$\frac{1}{21}(6, 15, 11, 21, 17, 27, 28)$	0.954	15	(4, 2)
3.22	✓	$\frac{1}{20}(7, 12, 10, 19, 17, 29, 24)$	0.909	15	(4, 2)
3.23	✓	$\frac{1}{11}(4, 5, 9, 9, 13, 14, 13)$	1.10	17	(4, 2)
3.24	✓	$\frac{1}{6}(5, 3, 4, 4, 8, 7, 7)$	1.50	22	(3)

TABLE 3. Classification of 7-dimensional indecomposable Einstein nilradicals. Rank three case.

Example 3.5. Applying Nikolayevsky's nice basis criterium

$\mathfrak{g}_{3.1(i_\lambda)}$

$$\begin{aligned} [e_1, e_2] &= e_4, [e_1, e_3] = e_5, [e_1, e_6] = e_7, [e_2, e_3] = e_6, [e_2, e_5] = \lambda e_7, \\ [e_3, e_4] &= (\lambda - 1)e_7 \end{aligned}$$

The basis $\{e_1, \dots, e_7\}$ is a nice basis to $\mathfrak{g}_{3.1(i_\lambda)}$ with any λ . We can use Theorem 2.8. If $\lambda \neq 0, 1$, the matrix U is given by

$$\begin{pmatrix} 3 & 1 & 1 & 1 & 1 & -1 \\ 1 & 3 & 1 & 1 & -1 & 1 \\ 1 & 1 & 3 & -1 & 1 & 1 \\ 1 & 1 & -1 & 3 & 1 & 1 \\ 1 & -1 & 1 & 1 & 3 & 1 \\ -1 & 1 & 1 & 1 & 1 & 3 \end{pmatrix}.$$

The general solution to the problem $Ux = [1]_6$ is give by

$$x = \left(t_2, t_1, \frac{1}{2} - t_1 - t_2, \frac{1}{2} - t_1 - t_2, t_1, t_2 \right)^T.$$

By taking t_1 and t_2 such that $0 < t_1 < \frac{1}{2}$, $t_2 < \frac{1}{2} - t_1$ we get a solution with positive coordinates. Hence, $\mathfrak{g}_{3.1(i_\lambda)}$ whit $\lambda \neq 0, 1$ is an Einstein Nilradical.

If $\lambda = 0$ the matrix U correspond to $\mathfrak{g}_{3.1(i_0)}$ is

$$\begin{pmatrix} 3 & 1 & 1 & 1 & -1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & -1 & 1 \\ 1 & 1 & -1 & 3 & 1 \\ -1 & 1 & 1 & 1 & 3 \end{pmatrix}.$$

The general solution to the problem $Ux = [1]_6$ is $(t, 0, \frac{1}{2} - t, \frac{1}{2} - t, t)^T$, and it follows from the nullity of the second coordinate, one obtains $\mathfrak{g}_{3.1(i_0)}$ is not an Einstein nilradical. By arguing analogously, one obtains $\mathfrak{g}_{3.1(i_1)}$ is not an Einstein nilradical.

Rank four					
n	EN	pre-Einstein derivation	Min	dim Der	dim DCS
4.1	✓	$\frac{1}{7}(4, 5, 4, 5, 9, 8, 9)$	1.4	20	(3)
4.2	✓	$\frac{2}{5}(1, 2, 2, 2, 3, 3, 3)$	1.67	25	(3)
4.3	✓	$\frac{1}{7}(5, 5, 6, 5, 4, 10, 9)$	1.4	21	(2)
4.4	✓	$\frac{4}{5}(1, 1, 1, 1, 1, 1, 2)$	1.67	28	(1)

TABLE 4. Classification of 7-dimensional indecomposable Einstein nilradicals. Rank four case.

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